

# $R$ -deformed Heisenberg algebra, anyons and $d = 2 + 1$ supersymmetry

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## Abstract

A universal minimal spinor set of linear differential equations describing anyons and ordinary integer and half-integer spin fields is constructed with the help of deformed Heisenberg algebra with reflection. The construction is generalized to some  $d = 2 + 1$  supersymmetric field systems. Quadratic and linear forms of action functionals are found for the universal minimal as well as for supersymmetric spinor sets of equations. A possibility of constructing a universal classical mechanical model for  $d = 2 + 1$  spin systems is discussed.

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## 1 Introduction

There are two different field-theoretical approaches to the description of anyons [1, 2, 3] as fundamental particle-like objects. One of them uses Chern-Simons gauge field whose role is to change spin and statistics of the charged matter field minimally coupled to it [3, 4, 5]. The initial formulation of the Chern-Simons construction has a local character, but the final gauge-invariant composite field carrying fractional spin and obeying fractional statistics turns out to be of an essentially nonlocal nature [4, 5]. Another is the so called group-theoretical approach [6]–[14], which aims at describing anyons by means of the methods applicable for ordinary integer and half-integer spin fields (in what follows, spin- $j$  fields). This approach can be formulated starting either from the construction of the appropriate field equations [7]–[13], or just from modelling fractional spin particles (further on, fractons) at the level of classical mechanics with a subsequent quantization of the theory [6, 9, 12]. However, it seems

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that spin- $j$  particles and fractons have quite different nature within the latter approach too. Indeed, spin degrees of freedom of spin- $j$  particles are described by anticommuting grassmann (odd) [15] or paragrassmann [16] variables in pseudoclassical mechanics, whereas fractons are described by constrained dynamics on extended “plain” phase space with commuting (even) spin variables [6, 9, 12]. Alternatively, fractional spin particles require using a monopole-like symplectic structure on the minimal “curved” phase space containing no spin variables at all [7, 12, 14]. At the level of field equations one uses *finite-dimensional non-unitary* representations of  $\overline{SO}(2, 1)$  for describing spin- $j$  fields, whereas *infinite-dimensional unitary* representations of the universal cover of  $d = 2 + 1$  Lorentz group have to be employed in the case of anyons<sup>1</sup> [7, 8, 10].

In this letter we shall reveal a general algebraic structure unifying the cases of spin- $j$  fields and fractons. Namely, we shall construct a universal covariant set of linear differential equations describing either spin- $j$  fields or fractional spin fields. This underlying algebraic structure will be given by the deformed Heisenberg algebra involving the reflection operator  $R$  [17]. Here we shall use the universality of the  $R$ -deformed Heisenberg algebra (RDHA) which has been established recently [18]: this algebra is related simultaneously to (generalized) parabosons and to (deformed) parafermions supplying us with irreducible unitary infinite-dimensional and non-unitary finite-dimensional representations of  $osp(1|2)$  superalgebra. The set of equations which will be constructed is a minimal spinor set of equations  $\mathcal{Q}_\alpha \Psi = 0$  with linear (in  $\partial_\mu$ ) differential spinor operator  $\mathcal{Q}_\alpha$  to be realized via the generators of RDHA. The hidden  $osp(1|2)$  superalgebraic structure of RDHA will also allow us to construct spinor linear sets of equations describing some  $(2+1)$ -dimensional supersymmetric systems.

The paper is organized as follows. In section 2 we formulate the problem of constructing a minimal universal covariant set of linear differential equations and describe a universal  $osp(1|2)$  superalgebraic hidden structure of RDHA necessary for realizing the construction. In section 3 we solve the formulated problem. Section 4 is devoted to generalization for the case of supersymmetric spinor sets of linear differential equations. Quadratic and linear variants of field actions giving rise to corresponding universal or SUSY sets of equations are presented in section 5. Section 6 is devoted to concluding remarks. A short Appendix contains some relations necessary for the constructions realized in the main text of the paper.

## 2 R-deformed Heisenberg algebra and $osp(1|2)$

In 2+1 dimensions, the fields  $\Psi = \Psi^n(x)$  obeying two equations

$$(P^2 + m^2)\Psi = 0, \quad (PJ - sm)\Psi = 0 \quad (2.1)$$

describe one-particle states realizing irreducible representations of  $d = 2 + 1$  quantum mechanical Poincaré group [2, 7, 10]. Here  $P_\mu = -i\partial_\mu$ ,  $\mu = 0, 1, 2$ ,  $J_\mu$  is translation-invariant part of the total angular momentum vector operator  $M_\mu = -\epsilon_{\mu\nu\lambda}x^\nu P^\lambda + J_\mu$ ,  $[J_\mu, J_\nu] = -i\epsilon_{\mu\nu\lambda}J^\lambda$ , and we use the metric  $\eta_{\mu\nu} = \text{diag}(-, +, +)$  and the totally antisymmetric tensor  $\epsilon_{\mu\nu\lambda}$ ,  $\epsilon^{012} = 1$ . Quadratic Klein-Gordon equation is the consequence of the

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<sup>1</sup>Using the term “anyons”, it is necessary to have in mind that the spin-statistics relation has not yet been proved for fractional spin particles within a group-theoretical approach.

second linear spin equation only under the choice of two-dimensional spinor ( $n = 1, 2$ ,  $s = \pm 1/2$ ) or three-dimensional vector ( $n = 1, 2, 3$ ,  $s = \pm 1$ ) representations. For any other finite-dimensional or any infinite-dimensional representation of  $so(2, 1)$ , these equations fix independently two Casimir operators of  $d = 2 + 1$  Poincaré group [7, 10]. Due to their independence, equations (2.1) are not very suitable as a starting point for the construction of corresponding field action and subsequent quantization of fractons and spin- $j$  fields with  $j > 1$  [7, 10, 13].

Here we shall use the Dirac's idea to generate Klein-Gordon equation as an integrability condition of some covariant set of linear differential equations [19]. Recently, such an idea was employed to describe fractons as well as  $d = 2 + 1$  spin- $j$  fields [9, 10, 11, 13]. However, the previous variants of linear differential equations describe either only fractons [9, 11, 13], or both  $d = 2 + 1$  spin- $j$  fields and anyons, but in a non-minimal way [10]. A spinor set of two linear differential equations to be constructed here will generate both equations (2.1) for the case of any representation of  $so(2, 1)$ . As a consequence, it will give us a universal minimal covariant description of fractons and spin- $j$  fields. Since the construction will be based on  $osp(1|2)$  superalgebra realized universally in terms of the  $R$ -deformed Heisenberg algebra generators, for the sake of completeness of exposition we describe briefly representations of the latter algebra and related realization of  $osp(1|2)$  generators.

The deformed Heisenberg algebra with reflection [17, 18] is given by the generators  $a^-$ ,  $a^+$ ,  $R$ , and 1, which satisfy the (anti)commutation relations

$$[a^-, a^+] = 1 + \nu R, \quad R^2 = 1, \quad \{a^\pm, R\} = 0, \quad [1, a^\pm] = [1, R] = 0, \quad (2.2)$$

where  $\nu \in \mathbf{R}$  is a deformation parameter and  $R$  is a reflection operator. In the case  $\nu > -1$  algebra (2.2) has infinite-dimensional unitary representations which can be realized on the Fock space with complete orthonormal basis of states  $|n\rangle = C_n(a^+)^n|0\rangle$ ,  $n = 0, 1, \dots$ ,  $a^-|0\rangle = 0$ ,  $\langle 0|0\rangle = 1$ ,  $R|0\rangle = |0\rangle$ , where  $C_n = ([n]_\nu!)^{-1/2}$ ,  $[n]_\nu! = \prod_{l=1}^n [l]_\nu$ ,  $[l]_\nu = l + \frac{1}{2}(1 - (-1)^l)\nu$ . The reflection operator acts as  $R|n\rangle = (-1)^n|n\rangle$ , introducing  $Z_2$ -grading structure in the space of states,  $R|k\rangle_\pm = \pm|k\rangle_\pm$ ,  $|k\rangle_+ = |2k\rangle$ ,  $|k\rangle_- = |2k+1\rangle$ ,  $k = 0, 1, \dots$ . The even, '+', and odd, '-', subspaces can be singled out by the projectors  $\Pi_\pm = \frac{1}{2}(1 \pm R)$ ,  $\Pi_\pm^2 = \Pi_\pm$ ,  $\Pi_+\Pi_- = 0$ ,  $\Pi_+ + \Pi_- = 1$ .

Algebra (2.2) has also  $(2r+1)$ -dimensional representations for the values of deformation parameter  $\nu = -(2r+1)$ ,  $r = 1, 2, \dots$  [18]. These representations are characterized by the relations  $a^{-(2r+1)} = a^{+(2r+1)} = 0$  being specific for paragrassmann algebras [20]. They can be realized as matrix representations with diagonal operator  $R = \text{diag}(+1, -1, +1, \dots, -1, +1)$ , and with operators  $a^\pm$  realized as  $(a^+)_{ij} = A_j\delta_{i-1,j}$ ,  $(a^-)_{ij} = B_i\delta_{i+1,j}$ , where  $A_{2k+1} = -B_{2k+1} = \sqrt{2(r-k)}$ ,  $k = 0, 1, \dots, r-1$ ,  $A_{2k} = B_{2k} = \sqrt{2k}$ ,  $k = 1, \dots, r$ . Operators  $a^+$  and  $a^-$  are conjugate,  $(\Psi_1, a^-\Psi_2)^* = (\Psi_2, a^+\Psi_1)$ , with respect to the indefinite scalar product given by

$$(\Psi_1, \Psi_2) = \bar{\Psi}_1 \Psi_2, \quad \bar{\Psi}_n = \Psi_n^* \hat{\eta}_{kn}, \quad (2.3)$$

where  $\Psi^n = \langle n|\Psi\rangle$  and  $\hat{\eta} = \text{diag}(1, -1, -1, +1, +1, \dots, (-1)^{r-1}, (-1)^{r-1}, (-1)^r, (-1)^r)$  is indefinite metric operator. Linear combinations  $\mathcal{L}_1 = \frac{1}{\sqrt{2}}(a^+ + a^-)$ ,  $\mathcal{L}_2 = \frac{i}{\sqrt{2}}(a^+ - a^-)$ , being the  $R$ -deformed coordinate and momentum operators,  $[\mathcal{L}_\alpha, \mathcal{L}_\beta] = i\epsilon_{\alpha\beta}(1 + \nu R)$ ,  $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ ,  $\epsilon_{12} = 1$ , are hermitian in the case of infinite-dimensional representations and self-conjugate with respect to scalar product (2.3) in the case of finite dimensional representations of RDHA.

Operators  $\mathcal{L}_\alpha$  together with quadratic operators  $J_\mu$ ,  $J_0 \equiv \frac{1}{4}\{a^+, a^-\}$ ,  $J_1 \pm iJ_2 = J_\pm = \frac{1}{2}(a^\pm)^2$ , form the set of generators of  $osp(1|2)$  superalgebra:  $\{\mathcal{L}_\alpha, \mathcal{L}_\beta\} = 4i(J\gamma)_{\alpha\beta}$ ,  $[J_\mu, J_\nu] = -i\epsilon_{\mu\nu\lambda}J^\lambda$ ,  $[J_\mu, \mathcal{L}_\alpha] = \frac{1}{2}(\gamma_\mu)_\alpha^\beta \mathcal{L}_\beta$ . These (anti)commutation relations mean that  $J_\mu$  are even and  $\mathcal{L}_\alpha$  are odd generators of the superalgebra and that the components of  $\mathcal{L}_\alpha$  form  $so(2, 1)$  spinor. Here  $\gamma$ -matrices, satisfying the relations  $\gamma^\mu\gamma^\nu = -\eta^{\mu\nu} + i\epsilon^{\mu\nu\lambda}\gamma_\lambda$ , appear in the Majorana representation,  $(\gamma^0)_\alpha^\beta = -(\sigma^2)_\alpha^\beta$ ,  $(\gamma^1)_\alpha^\beta = i(\sigma^1)_\alpha^\beta$ ,  $(\gamma^2)_\alpha^\beta = i(\sigma^3)_\alpha^\beta$ , and so, they satisfy also the relations  $\gamma_{\alpha\beta}^{\mu\dagger} = -\gamma_{\alpha\beta}^\mu$ ,  $\gamma_{\alpha\beta}^\mu = \gamma_{\beta\alpha}^\mu$ . Raising and lowering spinor indices is realized by  $\epsilon_{\alpha\beta}$  and by the antisymmetric tensor  $\epsilon^{\alpha\beta}$ ,  $\epsilon^{12} = 1$ :  $f_\alpha = f^\beta\epsilon_{\beta\alpha}$ ,  $f^\alpha = \epsilon^{\alpha\beta}f_\beta$ . The  $osp(1|2)$  Casimir operator  $\mathcal{C} \equiv J^\mu J_\mu - \frac{i}{8}\mathcal{L}^\alpha\mathcal{L}_\alpha$  takes the fixed value  $\mathcal{C} = \frac{1}{16}(1 - \nu^2)$  and therefore, every infinite- or finite-dimensional representation of RDHA supplies us with corresponding irreducible representation of  $osp(1|2)$  superalgebra.

On the other hand, every such representation is reducible with respect to the action of the  $so(2, 1) \sim sl(2, R)$  generators  $J_\mu$ :  $J^2 = J^\mu J_\mu = -\hat{\alpha}(\hat{\alpha} - 1)$ , where  $\hat{\alpha} = \frac{1}{4}(1 + \nu R)$ . Therefore,  $J_\mu$  act irreducibly on even, '+', and odd, '-', subspaces spanned by the states  $|k\rangle_+$  and  $|k\rangle_-$ ,  $J^2|k\rangle_\pm = -\alpha_\pm(\alpha_\pm - 1)|k\rangle_\pm$ , where  $\alpha_+ = \frac{1}{4}(1 + \nu)$ ,  $\alpha_- = \alpha_+ + 1/2$  and  $J_0|k\rangle_\pm = (\alpha_\pm + k)|k\rangle_\pm$ ,  $k = 0, 1, \dots$ . In the case of infinite-dimensional representations of RDHA ( $\nu > -1$ ), this gives the direct sum of infinite-dimensional unitary irreducible representations of  $sl(2, R)$ ,  $\mathcal{D}_{\alpha_+}^+ \oplus \mathcal{D}_{\alpha_-}^+$ , being representations of the so called discrete series with parameters  $\alpha_+ > 0$  and  $\alpha_- > 1/2$  [13]. In the case of finite-dimensional representations of RDHA, we have the relations  $J^2|l\rangle_\pm = -j_\pm(j_\pm + 1)|l\rangle_\pm$ ,  $j_+ = r/2$ ,  $j_- = (r - 1)/2$ , where  $l = 0, 1, \dots, r$  for  $|l\rangle_+$  and  $l = 0, 1, \dots, r - 1$  for  $|l\rangle_-$ . Therefore, we have the direct sum of spin- $j_+$  and spin- $j_-$  finite-dimensional representations with  $so(2, 1)$  spin parameter shifted in  $1/2$ , where the operator  $J_0$  has the spectra  $j_0 = (-j_+, -j_+ + 1, \dots, j_+)$  and  $j_0 = (-j_-, -j_- + 1, \dots, j_-)$ , respectively [18].

As we shall see, the presence of two irreducible representations of  $so(2, 1)$  with corresponding spin parameter shifted in  $1/2$  will allow us to realize (2+1)-dimensional SUSY.

### 3 Universal spinor set of equations

Let us supplement  $d = 2 + 1$  Lorentz spinor operator  $\mathcal{L}_\alpha$  with the set of spinor operators  $\mathcal{P}_\alpha \equiv (P\gamma)_\alpha^\beta \mathcal{L}_\beta$ ,  $\mathcal{J}_\alpha \equiv i\mathcal{L}_\beta \epsilon_{\mu\nu\lambda} P^\mu J^\nu (\gamma^\lambda)_\alpha^\beta$  and  $\mathcal{L}_\alpha(PJ)$  linearly depending on  $P_\mu$ . Due to the identity relation  $\mathcal{L}_\alpha(PJ) + \frac{1}{4}(3 + \nu R)\mathcal{P}_\alpha - \mathcal{J}_\alpha \equiv 0$ , only any two operators from the introduced  $P_\mu$ -dependent set can be considered as independent spinor operators. We shall use the first two operators,  $\mathcal{P}_\alpha$  and  $\mathcal{J}_\alpha$ , as independent ones. The introduced operators have the properties  $\mathcal{P}_\alpha^\dagger = -\mathcal{P}_\alpha$ ,  $\mathcal{J}_\alpha^\dagger = \mathcal{J}_\alpha - \mathcal{P}_\alpha$ , where we mean the conjugation with respect to the positive definite scalar product  $\langle \Psi_1, \Psi_2 \rangle = \Psi_1^{*n} \Psi_2^n$  in the case  $\nu > -1$ , and with respect to indefinite scalar product (2.3) for  $\nu = -(2r + 1)$ ,  $r = 1, 2, \dots$

One considers the self-conjugate spinor operator

$$\mathcal{Q}_\alpha = \mathcal{Q}_\alpha^\dagger = R\mathcal{P}_\alpha + \epsilon m\mathcal{L}_\alpha, \quad \epsilon = \pm 1, \quad (3.1)$$

with the following (anti)commutation properties:

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = 4i(P^2 + m^2)(J\gamma)_{\alpha\beta} - 8i(\Delta_+\Pi_+ + \Delta_-\Pi_-)(P\gamma)_{\alpha\beta}, \quad (3.2)$$

$$[\mathcal{Q}_\alpha, \mathcal{Q}_\beta] = -(\mathcal{Q}^\rho \mathcal{Q}_\rho)\epsilon_{\alpha\beta}, \quad \mathcal{Q}^\rho \mathcal{Q}_\rho = i(P^2 + m^2)(1 + \nu R) + 8i\epsilon m(\Delta_+\Pi_+ - \Delta_-\Pi_-), \quad (3.3)$$

where  $\Delta_{\pm} = PJ - \epsilon m \frac{1}{4}(\nu \pm 1)$ . The list of relations has been used for calculating eqs. (3.2), (3.3) is given in Appendix. One introduces the field  $\Psi = \Psi^n(x)$ , which carries the corresponding irreducible representation of RDHA and is transformed appropriately under the action of the Lorentz transformations specified by parameters  $\omega_{\mu}$ ,  $\Psi(x) \rightarrow \Psi'(x') = \exp(iM_{\mu}\omega^{\mu})\Psi(x)$ . Now, let us postulate the spinor set of equations

$$\mathcal{Q}_{\alpha}\Psi = 0. \quad (3.4)$$

If the field  $\Psi(x)$  satisfy equations (3.4), in the Lorentz-transformed system it will satisfy the equations  $\mathcal{Q}'_{\alpha}\Psi'(x') = 0$ ,  $\mathcal{Q}'_{\alpha} = \exp(iM^{\mu}\omega_{\mu})\mathcal{Q}_{\alpha}\exp(-iM^{\mu}\omega_{\mu})$ . Therefore, the theory will have a covariant content. Due to eqs. (3.2), (3.3), we find that the fields  $\Psi_{\pm}$ ,  $\Psi = \Psi_{+} + \Psi_{-}$ ,  $R\Psi_{\pm} = \pm\Psi_{\pm}$ , satisfying eqs. (3.4), will satisfy also quadratic Klein-Gordon and linear spin equation of the form (2.1) with parameter  $s_{+} = \epsilon\frac{1}{4}(\nu + 1)$  for the case of field  $\Psi_{+}$  and  $s_{-} = \epsilon\frac{1}{4}(\nu - 1)$  for the field  $\Psi_{-}$ . Then we can go over to the momentum representation and choose the rest frame system  $P^{\mu} = (\epsilon^0 m, 0, 0)$ ,  $|\epsilon^0| = 1$ . As a result, we find that the second equation from (2.1) has only trivial solution in the ‘-’ subspace, i.e.  $\Psi_{-} = 0$ . On the other hand, one finds that the second equation has nontrivial solutions in the ‘+’ subspace with  $\epsilon^0$  taking values  $+1$  and  $-1$  in the case of finite-dimensional representations and taking only the value  $+1$  when  $\nu > -1$ . The corresponding solution  $\Psi_{+}$  describes irreducible representations of the Poincaré group with  $P^2 = -m^2$  and spin  $s_{+} = \epsilon\frac{1}{4}(1 + \nu)$ . The presence of the discrete parameter  $\epsilon$  in operator (3.1) allows us to describe the states with spin of any sign.

Let us discuss the obtained result in more detail. As we have seen, equations (2.1) appear as a consequence of equations (3.4). For the choice of finite dimensional representations of deformed Heisenberg algebra corresponding to the values of deformation parameter  $\nu = -3$  and  $\nu = -5$ , the second equation from (2.1) taken for corresponding nontrivial field  $\Psi_{+}$  is nothing else as the Dirac or Deser-Jackiw-Templeton-Schonfeld (DJTS) equation for the topologically massive vector gauge field [21]. Therefore, we conclude that the constructed set of equations (3.4) for the choice of the parameter  $\nu = -3$  and  $\nu = -5$  is equivalent either to the Dirac equation or to the DJTS equation. For other choices of finite-dimensional representations of RDHA ( $\nu = -(2r + 1)$ ,  $r = 3, 4, \dots$ ), the basic set of equations (3.4) describes irreducible representations of the  $d = 2 + 1$  Poincaré group with integer,  $s_{+} = -\epsilon k$  (for  $r = 2k$ ), and half integer,  $s_{+} = -\epsilon(k + 1/2)$  (for  $r = 2k + 1$ ), spin values. For infinite-dimensional representations of RDHA, our spinor set (3.4) is some (2+1)-dimensional analog of the Dirac positive-energy set of equations [19] related to the infinite-component Majorana positive-energy equation [22, 8]. Indeed, in the momentum representation we get the positive energy solution  $\Psi_{+}$  carrying arbitrary nonzero spin  $s_{+} = \epsilon\alpha_{+}$ ,  $\alpha_{+} = \frac{1}{4}(1 + \nu)$ ,

$$\Psi_{+\mathbf{k}}^n(P) = (-1)^n C_n^{1/2} \left( \frac{P_1 - iP_2}{P^0 + m} \right)^n \Psi_{+\mathbf{k}}^0(P), \quad \Psi_{+\mathbf{k}}^0(P) = \delta(P^0 - \sqrt{\mathbf{P}^2 + m^2})\delta(\mathbf{P} - \mathbf{k})f(\mathbf{k}), \quad (3.5)$$

where  $C_n = \Gamma(2\alpha_{+} + n)/\Gamma(2\alpha_{+})\Gamma(n + 1)$  and  $f(\mathbf{k})$  is an arbitrary function.

Therefore, the constructed minimal spinor set of linear differential equations (3.4) has a universal character describing spin- $j$  fields and anyons.

On the even subspace ‘+’ spanned by the states of the form  $|k\rangle_{+} \propto (a^{+})^{2k}|0\rangle$ , the following relation takes place:  $-\frac{i}{4}\mathcal{L}^{\alpha}(\gamma_{\mu})_{\alpha\beta}\mathcal{Q}^{\beta} = \frac{1}{4}(1 + \nu)P_{\mu} + \epsilon m J_{\mu} - i\epsilon_{\mu\nu\lambda}P^{\nu}J^{\lambda} \equiv V_{\mu}$ . Operator  $V_{\mu}$  is exactly the operator generating the universal vector set of linear differential equations

$V_\mu \Psi = 0$  constructed in ref. [10]. However, due to the operator identity  $T^\mu V_\mu \equiv 0$ ,  $T_\mu = (\frac{1}{4}(\alpha_+ - 1)^2 \eta_{\mu\nu} - i(\alpha_+ - 1) \epsilon_{\mu\nu\lambda} J^\lambda + J_\nu J_\mu) P^\nu$ , the vector set, unlike the present spinor case, is the set of dependent equations. This fact made the problem of construction of the corresponding field action to be complicated [10].

The spinor operator  $\mathcal{Q}_\alpha$ , unlike the operator  $\mathcal{D}_\alpha = \epsilon m \mathcal{L}_\alpha - \mathcal{P}_\alpha \neq \mathcal{D}_\alpha^\dagger$  used earlier for constructing the minimal set of anyonic equations [11, 13], is self-conjugate,  $\mathcal{Q}_\alpha^\dagger = \mathcal{Q}_\alpha$ . As we shall see, this property of our basic spinor operator will be important for formulating the problem of construction of the classical spin model corresponding to the universal field system (3.4).

## 4 (2+1)-dimensional SUSY

Any irreducible representation of the constructed  $osp(1|2)$  superalgebra contains the direct sum of two irreducible representations of  $so(2, 1)$  subalgebra, which are specified by the parameters  $\alpha_+$  and  $\alpha_-$  or  $j_+$  and  $j_-$  related as  $\alpha_- - \alpha_+ = j_+ - j_- = 1/2$ . Therefore, it seems rather natural to try to modify the set of linear differential equations in such a way that it would have nontrivial solutions not only in ‘+’ subspace, but also in ‘-’ subspace. If these states will have equal mass but their spin will be shifted for  $\Delta s = 1/2 \pmod{n}$ , we shall have the states forming a supermultiplet. As we shall see, it is indeed possible to find such a modification of spinor set of linear differential equations, but a program turns out to be realizable only for some special cases of finite-dimensional representations of RDHA.

In order to realize such a modification, let us consider the spinor set of equations

$$\mathcal{D}_\alpha \Psi = 0 \quad (4.1)$$

with spinor linear differential operator of the most general form,  $\mathcal{D}_\alpha = \mathcal{D}_\alpha^+ \Pi_+ + \mathcal{D}_\alpha^- \Pi_-$ , where  $\mathcal{D}_\alpha^\pm = \mathcal{P}_\alpha A^\pm + \mathcal{J}_\alpha B^\pm - m \mathcal{L}_\alpha C^\pm$ , and  $A^\pm$ ,  $B^\pm$  and  $C^\pm$  are  $c$ -number constants. Contracting equations (4.1) with  $\mathcal{L}^\alpha$  (see Appendix), we get

$$\left( PJ(A^\pm + B^\pm) - m \frac{1}{4} (1 \pm \nu) C^\pm \right) \Psi_\pm = 0. \quad (4.2)$$

In order this would give spin equations  $(PJ - ms_\pm) \Psi_\pm = 0$ , one has to have  $A^\pm + B^\pm \neq 0$  and  $\frac{1}{4} (1 \pm \nu) C^\pm = s_\pm (A^\pm + B^\pm)$ . Then, the contraction of eq. (4.1) with  $\mathcal{P}^\alpha$  gives

$$F^\pm \left( P^2 + m^2 \frac{16s_\pm^2}{(1 \pm \nu)^2} \right) \Psi_\pm = 0, \quad F^\pm = A^\pm + \frac{1}{4} (3 \mp \nu) B^\pm. \quad (4.3)$$

At last, the contraction with  $\mathcal{J}^\alpha$  gives conditions (4.3) multiplied by  $(1 \pm \nu)^2$ . If  $F^+ \neq 0$  and  $A^+ + B^+ \neq 0$ , two conditions (4.2) and (4.3) are consistent on the ‘+’ subspace for  $|s_+| = \frac{1}{4} (1 + \nu)$ , and as a consequence, for  $\Psi_+$  we have the Klein-Gordon and spin equations of the form (2.1). On the other hand if  $F^- \neq 0$ , corresponding conditions for ‘-’ subspace have only trivial solution  $\Psi_- = 0$ . Let  $F^- = 0$ ,  $B^- \neq 0$  and  $C^- = s_- B_-$ . Then we have  $\mathcal{D}_\alpha^+ = \mathcal{P}_\alpha A^+ + \mathcal{J}_\alpha B^+ + \epsilon m \mathcal{L}_\alpha (A^+ + B^+)$ ,  $\epsilon = \pm 1$ , and  $\mathcal{D}_\alpha^- = (-\frac{1}{4} (3 + \nu) \mathcal{P}_\alpha + \mathcal{J}_\alpha - m s_- \mathcal{L}_\alpha) B^-$ . As a result, in this case eqs. (4.1) lead to equations (2.1) for the ‘+’ subspace, whereas for ‘-’ subspace we have only one equation,  $(PJ - m s_-) \Psi_- = 0$ . This equation itself generally

does not fix the value of the Casimir operator  $P^2$  neither for the case  $\nu > -1$  nor for the case  $\nu = -(2r+1)$ . However, there are two exceptional cases given by  $\nu = -5$  and  $\nu = -7$ . In these two cases the mass shell condition  $(P^2 + m^2)\Psi_- = 0$  appears on the ‘-’ subspace as a consequence of the spin equation being the Dirac or DJTS equation.

Let us require that the spinor operator  $\mathcal{D}_\alpha$  would be self-conjugate, i.e.  $\mathcal{D}_\alpha^\dagger = \mathcal{D}_\alpha$ . This requirement leads to the conditions  $(B^-)^* = B^+$ ,  $A^+ = -\frac{1}{4}(1-\nu)B^+$ , and to the relation  $s_- = s_+(3+\nu)/(1+\nu)$ . In the case  $\nu = -5$ , we get  $s_+ = \epsilon$ ,  $s_- = \frac{1}{2}s_+$ , i.e. we have a massive supermultiplet of states with the spin content  $(s_+ = \epsilon, s_- = \frac{1}{2}\epsilon)$ . For  $\nu = -7$ , we have a supermultiplet with the spin content  $(s_+ = \frac{3}{2}\epsilon, s_- = \epsilon)$ . Putting  $B^- = -1$ , for the cases  $\nu = -5$  ( $r = 2$ ) and  $\nu = -7$  ( $r = 3$ ), we have equation (4.1) with

$$\mathcal{D}_\alpha = \frac{r-1}{2}\epsilon m\mathcal{L}_\alpha - \mathcal{J}_\alpha + \frac{1}{2}(1-rR)\mathcal{P}_\alpha, \quad r = 2, 3. \quad (4.4)$$

It is interesting to note that in both cases,  $r = 2, 3$ , the spinor supercharge operator  $\mathcal{Q}_\alpha$ ,  $\{\mathcal{Q}_\alpha, \mathcal{D}_\beta\} \approx 0$ , which transforms the corresponding supermultiplet components, coincides with operator (3.1). The weak equality means that the left hand side turns into zero on the physical subspace given by eqs. (4.1), and we have the following typical superalgebraic relations:  $\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} \approx -4i\epsilon m(r+2)(P\gamma)_{\alpha\beta}$ ,  $[P_\mu, \mathcal{Q}_\alpha] = 0$ .

If we not require that  $\mathcal{D}_\alpha^\dagger = \mathcal{D}_\alpha$ , we shall have a possibility to get supermultiplets with exotic spin content  $(s_+ = \epsilon, s_- = -\frac{1}{2}\epsilon)$  and with corresponding spin shift  $|\Delta s| = |s_+ - s_-| = 3/2$  for  $\nu = -5$ , and  $(s_+ = \frac{3}{2}\epsilon, s_- = -\epsilon)$  with  $|\Delta s| = 5/2$  for  $\nu = -7$ . Choosing  $B^+ = B^- = -1$ ,  $A^+ = 1/2$  for  $\nu = -5$ , and  $B^+ = B^- = -1/2$ ,  $A^+ = 0$  for  $\nu = -7$ , we shall obtain

$$\mathcal{D}_\alpha = -\frac{1}{2}\epsilon m\mathcal{L}_\alpha - \mathcal{J}_\alpha - \frac{1}{2}R\mathcal{P}_\alpha, \quad \nu = -5; \quad \mathcal{D}_\alpha = -\frac{1}{2}\epsilon m\mathcal{L}_\alpha - \frac{1}{2}\mathcal{J}_\alpha - \frac{1}{4}(1+R)\mathcal{P}_\alpha, \quad \nu = -7. \quad (4.5)$$

Both these operators have the property  $\mathcal{D}_\alpha^\dagger = \mathcal{D}_\alpha + \mathcal{P}_\alpha$ .

## 5 Field action

Since universal equations (3.4) are two equations for one field  $\Psi(x)$ , the corresponding field action giving rise to them has to contain some auxiliary fields. Let us consider the action

$$S = \int L d^3x \quad (5.1)$$

with Lagrangian  $L = \bar{\chi}^\alpha \mathcal{Q}_\alpha \mathcal{Q}_\beta \chi^\beta + \bar{\chi}^\alpha \mathcal{Q}_\alpha \Psi + \bar{\Psi} \mathcal{Q}_\alpha \chi^\alpha + c\bar{\Psi}\Psi$ , containing auxiliary field  $\chi^\alpha$  with dimensionality  $[\chi^\alpha] = m^{1/2}$ , whereas  $[\Psi] = m^{3/2}$ , and here  $c = c^*$  is a real dimensionless constant. The conjugate fields are  $\bar{\Psi} = \Psi^\dagger$ ,  $\bar{\chi}^\alpha = \chi^{\dagger\alpha}$  for  $\nu > -1$ , whereas  $\bar{\Psi} = \Psi^\dagger \hat{\eta}$ ,  $\bar{\chi}^\alpha = \chi^{\dagger\alpha} \hat{\eta}$  for  $\nu = -(2r+1)$  that guarantees the reality of the Lagrangian. Action (5.1) leads to the equations  $\mathcal{Q}_\alpha \mathcal{Q}_\beta \chi^\beta + \mathcal{Q}_\alpha \Psi = 0$ ,  $\mathcal{Q}_\alpha \chi^\alpha + c\Psi = 0$ , and to the corresponding equations for the conjugate fields  $\bar{\Psi}$  and  $\bar{\chi}^\alpha$ . Substituting the second equation into the first, we arrive at the equivalent system of equations  $\mathcal{Q}_\alpha(1-c)\Psi = 0$ ,  $\mathcal{Q}_\alpha \chi^\alpha + c\Psi = 0$ . Therefore, if  $c \neq 1$ , we have the necessary spinor set of equations (3.4) for the basic field  $\Psi$ . Now, we have to arrange the construction in such a way that the field  $\chi^\alpha$  would be a pure auxiliary field

having no independent dynamics. To do this, we decompose the field  $\chi^\alpha$  into “longitudinal” and “transverse” parts,  $\chi^\alpha = \mathcal{Q}^\alpha \chi_l + \chi_\perp^\alpha$ , where  $\mathcal{Q}_\alpha \chi^\alpha = \chi_l$ , and  $\mathcal{Q}_\alpha \chi_\perp^\alpha = 0$ . Due to the equations of motion, the “longitudinal” part of  $\chi^\alpha$  coincides up to a  $c$ -number factor with  $\Psi$ . If we choose  $c = 0$ , the “longitudinal” part of  $\chi^\alpha$  will not contribute at all to the theory. On the other hand, the “transverse” part of  $\chi^\alpha$  is a pure gauge degree of freedom of the theory due to the invariance of the action and of the equations of motion with respect to the transformations  $\chi^\alpha \rightarrow \chi^\alpha + \Pi^{\alpha\beta} \Lambda_\beta$ , where  $\Pi_{\alpha\beta} = (\mathcal{Q}^\sigma \mathcal{Q}_\sigma) \epsilon_{\alpha\beta} - \mathcal{Q}_\alpha \mathcal{Q}_\beta$ ,  $\mathcal{Q}^\alpha \Pi_{\alpha\beta} = 0$ . Therefore, at  $c = 0$ ,  $\chi^\alpha(x)$  plays a role of an auxiliary spinor field. Finally, the Lagrangian can be represented in the form

$$L = (\bar{\Psi} + \bar{\chi}^\alpha \mathcal{Q}_\alpha)(\Psi + \mathcal{Q}_\beta \chi^\beta) - \bar{\Psi} \Psi. \quad (5.2)$$

Note that the equation  $\mathcal{Q}_\alpha \chi^\alpha = 0$  taking place for the choice  $c = 0$  in the initial Lagrangian, together with the basic spinor set of linear differential equations (3.4) may be obtained from the action with the linear form of Lagrangian,

$$L' = \bar{\chi}^\alpha \mathcal{Q}_\alpha \Psi + \bar{\Psi} \mathcal{Q}_\alpha \chi^\alpha. \quad (5.3)$$

The change of  $\mathcal{Q}_\alpha$  for  $\mathcal{D}_\alpha$  in Lagrangians (5.2) and (5.3) will give the action functionals for the special SUSY cases (4.4) corresponding to  $\nu = -5$  and  $\nu = -7$ . For SUSY systems given by eqs. (4.1), (4.5), the corresponding quadratic and linear Lagrangians are  $L = (\bar{\Psi} + \bar{\chi}^\alpha \mathcal{D}_\alpha)(\Psi + \mathcal{D}_\beta^\dagger \chi^\beta) - \bar{\Psi} \Psi$  and  $L' = \bar{\chi}^\alpha \mathcal{D}_\alpha \Psi + \bar{\Psi} \mathcal{D}_\alpha^\dagger \chi^\alpha$ .

## 6 Concluding remarks

In the approach with Chern-Simons gauge field, a gauge-invariant anyonic composite field is a nonlocal field of the initial statistically charged matter field and it is given on a half-infinite non-observable space-like string [4, 5]. In the present approach, we describe fractons by infinite-component field with index  $n$  taking half-infinite set of values,  $n = 0, 1, \dots$ . At the same time, in accordance with eq. (3.5) this infinite-component field carries only one independent field degree of freedom. Therefore, there is some analogy of the group-theoretical approach with the Chern-Simons approach. The spin of fractons can be varied continuously by changing the deformation parameter  $\nu$ . In particular, it can take integer or half-integer values. However, in such cases the description of integer and half-integer spin fields is essentially different from the description of spin- $j$  fields of the same spin value: spin- $j$  fields are described by non-unitary finite-dimensional  $so(2, 1)$  representations, whereas fracton case corresponds to employing unitary infinite-dimensional representations. Perhaps, the above mentioned analogy means that the description of integer and half-integer relativistic spin fields given by the Chern-Simons approach under the appropriate choice of the statistical parameter is not reducible to the ordinary description of spin- $j$  fields (see, however, ref. [23]).

The quantization of the theory presented here deserves a detailed Hamiltonian analysis. As we noted, in the case of the choice of ‘anyonic representation’ of the  $R$ -deformed Heisenberg algebra ( $\nu > -1$ ), infinite-component field carries effectively only one independent physical degree of freedom. Therefore, corresponding Hamiltonian theory has to contain infinite set of constraints removing all the field degrees of freedom except one. Such infinite



set of constraints should appropriately be taken into account. The subsequent quantization of the theory given by Lagrangians (5.2) and (5.3) has to answer the question on the spin-statistics relation for fractons.

Our final remark concerns the construction of classical mechanical model which would correspond to the field system defined by eqs. (3.1), (3.4). Due to the property  $\mathcal{Q}_\alpha^\dagger = \mathcal{Q}_\alpha$ , one can interpret equations (3.4) as quantum analogs of some classical constraints  $q_\alpha \approx 0$  which should be supplemented by the constraints  $p^2 + m^2 \approx 0$  and  $pj - ms \approx 0$  being the classical analogs of the Klein-Gordon and spin equations. In this case reflection operator should be understood as the operator realized via creation-annihilation operators  $a^\pm$  as  $R = \cos \pi N$ ,  $N = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}(1 + \nu)$ . Therefore, in this case spinor operators (3.1) can be understood as essentially nonlinear operators in terms of operators  $a^\pm$ . Then, the problem will be reduced to the problem of constructing a classical analog of algebra (2.2) with  $R = \cos \pi N$ . It seems that the classical analog of this algebra can be constructed using the ideas of ref. [24] where the classical analog of the  $q$ -deformed oscillator was realized as a constrained system given on a Kähler manifold. The construction of such universal classical mechanical model for  $d = 2 + 1$  spin systems could help in resolving some open problems taking place in pseudoclassical approach to relativistic spin- $j$  particles [25].

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## A Appendix

The following contraction relations take place for the operators  $\mathcal{L}_\alpha$ ,  $\mathcal{P}_\alpha$  and  $\mathcal{J}_\alpha$ :

$$\begin{aligned}\mathcal{L}^\alpha \mathcal{L}_\alpha &= -i(1 + \nu R), \quad \mathcal{L}^\alpha \mathcal{P}_\alpha = -\mathcal{P}^\alpha \mathcal{L}_\alpha = \mathcal{L}^\alpha \mathcal{J}_\alpha = -4i(PJ), \quad \mathcal{J}^\alpha \mathcal{L}_\alpha = 0, \\ \mathcal{P}^\alpha \mathcal{P}_\alpha &= -i(1 + \nu R)P^2, \quad \mathcal{J}^\alpha \mathcal{P}_\alpha = 4i(P^2 J^2 - (PJ)^2) - i(1 + \nu R)P^2, \\ \mathcal{P}^\alpha \mathcal{J}_\alpha &= 4i((PJ)^2 - P^2 J^2), \quad \mathcal{J}^\alpha \mathcal{J}_\alpha = i(1 + \nu R)((PJ)^2 - P^2 J^2) - 4i(PJ)^2.\end{aligned}$$

The complete set of (anti)commutation relations for these operators is the following:

$$\begin{aligned}[\mathcal{S}_\alpha, \mathcal{S}_\beta] &= -(\mathcal{S}^\rho \mathcal{S}_\rho)\epsilon_{\alpha\beta}, \quad \mathcal{S}_\alpha = \mathcal{L}_\alpha, \mathcal{P}_\alpha, \mathcal{J}_\alpha, \\ \{\mathcal{L}_\alpha, \mathcal{L}_\beta\} &= 4i(J\gamma)_{\alpha\beta}, \quad \{\mathcal{P}_\alpha, \mathcal{P}_\beta\} = 8i(PJ)(P\gamma)_{\alpha\beta} - 4iP^2(J\gamma)_{\alpha\beta}, \\ [\mathcal{L}_\alpha, \mathcal{P}_\beta] &= -i(1 + \nu R)(P\gamma)_{\alpha\beta}, \quad \{\mathcal{L}_\alpha, \mathcal{P}_\beta\} = 4i(PJ)\epsilon_{\alpha\beta} - 4\Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} \equiv \epsilon_{\mu\nu\lambda} P^\mu J^\nu \gamma_{\alpha\beta}^\lambda, \\ [\mathcal{L}_\alpha, \mathcal{J}_\beta] &= -\frac{i}{2}(1 + \nu R)(P\gamma)_{\alpha\beta} + (1 + \nu R)\Gamma_{\alpha\beta} + 2i(PJ)\epsilon_{\alpha\beta}, \\ \{\mathcal{L}_\alpha, \mathcal{J}_\beta\} &= -4\Gamma_{\alpha\beta} + 2i(PJ)\epsilon_{\alpha\beta} + 4i(J\gamma)_{\alpha\beta}(PJ) + \frac{i}{4}(\nu^2 - 1)(P\gamma)_{\alpha\beta}, \\ [\mathcal{P}_\alpha, \mathcal{J}_\beta] &= \frac{i}{2}(1 + \nu R)P^2\epsilon_{\alpha\beta} + i(1 + \nu R)P^2(J\gamma)_{\alpha\beta} + i(1 - \nu R)(PJ)(P\gamma)_{\alpha\beta}, \\ \{\mathcal{P}_\alpha, \mathcal{J}_\beta\} &= -4iP^2(J\gamma)_{\alpha\beta} + 6i(PJ)(P\gamma)_{\alpha\beta} - 4\Gamma_{\alpha\beta}(PJ) + \frac{i}{4}((1 - \nu^2)P^2 - 16(PJ)^2)\epsilon_{\alpha\beta}, \\ \{\mathcal{J}_\alpha, \mathcal{J}_\beta\} &= 4i(J\gamma)_{\alpha\beta}\left((PJ)^2 - P^2 + \frac{1}{16}(1 + \nu R)^2 P^2\right) + 4i(PJ)(P\gamma)_{\alpha\beta} - 8\Gamma_{\alpha\beta}(PJ).\end{aligned}$$

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